Technical report #7

presented to

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17 July 1998

[1]Supported by the Ministry of Education of Japan

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Surge tank system of a hydro-electric power plant

A surge tank used system used in hydro-electric power plants is shown in Figure 1 [ATR74].

From Figure 1 the mass of the fluid in the pipe connecting the reservoir to the surge tank #1 is

$$m_{12} = \rho a_{12} l_{12}, \quad \mathrm{kg}, \tag{1}$$

where

$$\begin{aligned} a_{ij} &= \mathrm{cross-sectionalareaoftheconduit} i \leftrightarrow j, \quad \mathrm{m}^2 \\ l_{ij} &= \mathrm{lengthoftheconduit} i \leftrightarrow j, \quad \mathrm{m} \\ \rho &= \mathrm{fluidmassdensity}, \quad \frac{\mathrm{kg}}{\mathrm{m}^3}. \end{aligned}$$

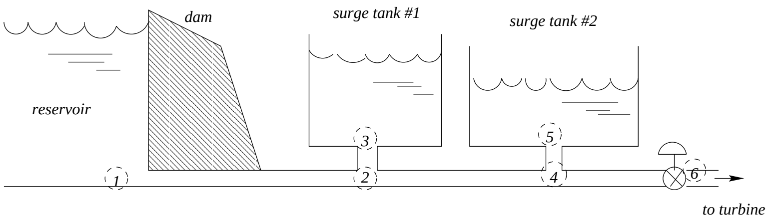

Figure 1: *Surge tank system for a hydro-electric power plant*

The force exerted on the mass m_{12} is

$$\frac{p_1 - p_2}{a_{12}}, \tag{2}$$

where p_i is the pressure at the point labelled i. A fluid velocity is

$$v_{12} = \frac{q_{12}}{a_{12}}, \quad \frac{\mathrm{m}}{\mathrm{sec}}, \tag{3}$$

where v_{ij} is the fluid velocity from point i to j.

At point labelled 3 we have

$$p_3 = \rho g h_1 = \rho g \frac{V_1}{A_1} = \frac{\rho g}{A_1} \int q_{23} dt, \quad \frac{\mathrm{N}}{\mathrm{m}^2} \tag{4}$$

where

$$\begin{array}{rcl} h_i & = & \mathrm{the height of water in the tank labelled} i, \quad \mathrm{m} \\ V_i & = & \mathrm{the liquid volume in the tank labelled} i, \quad \mathrm{m}^3 \\ A_i & = & \mathrm{the area of the tank labelled} i, \quad \mathrm{m}^2 \\ g & = & \mathrm{the local value of the acceleration of gravity}, \quad \frac{\mathrm{m}}{\mathrm{sec}^2}. \end{array}$$

From the law of continuity, at the junction labelled 2

$$q_{23} = q_{12} - q_{24}. \tag{5}$$

For the surge tank #2

$$p_5 = \rho g h_2 = \rho g \frac{V_2}{A_2} = \frac{\rho g}{A_2} \int q_{45} dt, \quad \frac{\mathrm{N}}{\mathrm{m}^2}. \tag{6}$$

The continuity condition at point 4 is

$$q_{45} = q_{24} - q_{46}. \tag{7}$$

p_1 is an independent variable and is the input of the system. For simplicity, assume that $p_6 = 0$.

The surge tank system can then be described by a set of four differential equations [ATR74]

$$\frac{q_{12}}{dt} = \frac{a_{12}}{\rho l_{12}}\left(p_1 - p_2\right) \tag{8}$$

$$\frac{p_3}{dt} = \frac{\rho g q_{23}}{A_1} \tag{9}$$

$$\frac{q_{24}}{dt} = \frac{a_{24}}{\rho l_{24}}\left(p_2 - p_4\right) \tag{10}$$

$$\frac{p_5}{dt} = \frac{\rho g q_{45}}{A_2} \tag{11}$$

The turbulent flow through a restriction connecting surge tank #1 with the main line, when the possibility of flow reversals is accounted for, is given by

$$q_{23} = \eta_1 \mathrm{sst}\left(p_2 - p_3\right), \tag{12}$$

where

$$\mathrm{sst}(x) = \mathrm{sgn}(x)\sqrt{|x|}, \qquad \mathrm{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases},$$

and η_i is a semi-empirical coefficient for an orifice described by

$$q_{i-\mathrm{out}} = \eta i \sqrt{h_i}. \tag{13}$$

The resistor to surge tank #2 is

$$q_{45} = \eta_2 \mathrm{sst}\left(p_4 - p_5\right). \tag{14}$$

The resistor at the wicket valve (gate valve) is

$$q_{46} = \eta_w \mathrm{sst}\left(p_4 - p_5\right), \tag{15}$$

the valve openning is assumed to be kept constant.

Let the state variables be

$$x_1 = q_{12}, \quad x_2 = p_3, \quad x_3 = q_{24}, \quad x_4 = p_5.$$

Linearized system of the surge tanks

The linearized state equations of the surge tank system is [ATR74]

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} -R_{23} & -1 & R_{23} & 0 \\ 1 & 0 & -1 & 0 \\ R_{23} & 1 & -\left(R_{23} + \frac{R_{45}}{C}\right) & \frac{1}{C} \\ 0 & 0 & \frac{1}{C} & -\frac{1}{R_{46}C} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} p_1 \\ 0 \\ \frac{R_{45}}{CR_{46}} p_6 \\ \frac{1}{CR_{46}} p_6 \end{bmatrix}, \tag{16}$$

where

$$\gamma_1 = \left(\frac{a_{12}}{\rho l_{12}}\right), \quad \gamma_2 = \left(\frac{\rho g}{A_1}\right), \quad \gamma_3 = \left(\frac{a_{24}}{\rho l_{24}}\right), \quad \gamma_4 = \left(\frac{\rho g}{A_2}\right), \quad C = \left(\frac{1 + R_{45}}{R_{46}}\right),$$

and R_{ij} is an equivalent linear resistances of flow between the points i and j.

Nuclear reactor

Consider a simplified nuclear reactor ([Lef65], Appendix B) where

$$\dot{\eta} = k\eta, \tag{17}$$

where η is the mean neutron density of a nuclear reactor, the reactivity k is a function of the state of the reactor and is assumed to be a linear function of η and the temperatures $y_1, \ldots, y_n$ of various components of the reactor. Let

$$k = k_0 + c^T y - \rho\eta, \tag{18}$$

where k_0 is a scalar constant and $y^T = \left[\begin{array}{ccc} y_1 & \cdots & y_n \end{array}\right]$. Assuming that the heat transfer arises from conduction, then Newton's law of cooling gives

$$\dot{y} = Ay - b\eta, \tag{19}$$

where A is assumed to be nonsingular.

At a critical point

$$Ay - b\eta = 0, \quad \text{and} \quad \left(k_0 + c^T y - \rho\eta\right)\eta = 0. \tag{20}$$

When the reactor is shut down,

$$y_1 = 0, \quad \eta_1 = 0, \tag{21}$$

The only critical points at the steady state operating point is

$$y_2 = A^{-1}b\left(\rho - c^T A^{-1} b\right)^{-1} k_0, \quad \eta_2 = \left(\rho - c^T A_{-1} b\right)^{-1} k_0, \tag{22}$$

where $\rho - c^T A_{-1} b \neq 0$ and $\eta, \eta_2 \geq 0$.

Changing the origin to (y_2, η_2) implies the following changes in coordinates

$$\begin{array}{lll} x & = y - y_2 & = y - A^{-1}b\left(\rho - c^T A^{-1} b\right)^{-1} k_0 \\ \theta & = \eta - \eta_2 & \eta - \left(\rho - c^T A_{-1} b\right)^{-1} k_0. \end{array} \tag{23}$$

Consequently the kinetic equations are

$$\dot{x} = Ax - b\theta \tag{24}$$

$$\dot{\theta} = k\left(\theta + \eta_2\right) \tag{25}$$

$$k = c^T x - \rho\theta, \tag{26}$$

where $\eta = \theta + \eta_2 > 0$

Notice also that this equation may also be written as

$$\dot{x} = Ax - b\eta_2\left(e^\sigma - 1\right) \tag{27}$$

$$\dot{\sigma} = c^T x - \rho\eta_2\left(e^\sigma - 1\right), \tag{28}$$

with a change of the a state variable

$$\sigma = \log\left(\frac{\theta + \eta_2}{\eta_2}\right) \tag{29}$$

which is in the standard form for indirect control with $\phi(\sigma) = \eta_2\left(e^\sigma - 1\right)$.

Pendulum with vibration

Consider a suspended pendulum when the suspension point is subjected to vertical vibrations of small amplitude and high frequency. [Kha96] (Example 8.10) The pendulum is shown in Figure 2. Let the motion of the suspension point be described by $a\sin\omega t$, where a is the amplitude and ω

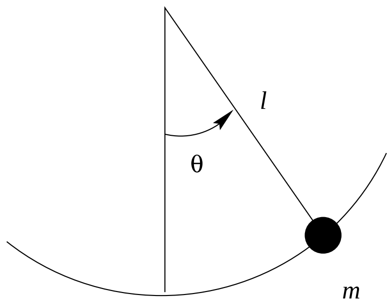

Figure 2: *Suspended pendulum*

the frequency. Then the coordinates of the bob are

$$x = l\sin\theta, \quad y = l\cos\theta - a\sin\omega t. \tag{30}$$

The velocity of the bob in the tangential direction is

$$l\dot{\theta} - a\omega\cos\omega t\sin\theta$$

and the acceleration in the tangential direction is

$$l\ddot{\theta} - a\omega^2\sin\omega t\sin\theta.$$

Assume that

$$\frac{a}{l} \ll 1, \quad \frac{\omega_0}{\omega} \ll 1,$$

where

$$\omega_0 = \sqrt{\frac{g}{l}}$$

is the frequency of free oscillation of the pendulum in the vicinity of the lower equilibrium position $\theta = 0$. The equation of motion in a new time scale $\tau = \omega t$ is

$$\frac{d^2\theta}{d\tau^2} + \alpha\beta\epsilon\frac{d\theta}{d\tau} + \left(\alpha^2\epsilon^2 - \epsilon\sin\tau\right)\sin\theta + \alpha\beta\epsilon^2\cos\tau\sin\theta = 0, \tag{31}$$

where

$$\epsilon = \frac{a}{l}, \quad \frac{\omega_0}{\omega} = \alpha\epsilon, \quad \alpha = \frac{\omega_0 l}{\omega a}, \quad \beta = \frac{k}{m\omega_0}.$$

The state equation is

$$\frac{dx}{d\tau} = \epsilon\begin{bmatrix} x_2 - \cos\tau\sin x_1 \\ -\alpha\beta x_2 - \alpha^2\sin x_1 + \cos\tau x_2\cos x_1 - \cos^2\tau\sin x_1\cos x_1 \end{bmatrix}, \tag{32}$$

where the state variables are

$$x_1 = \theta \tag{33}$$

$$x_2 = \frac{1}{\epsilon}\frac{d\theta}{d\tau} + \cos\tau\sin\theta \tag{34}$$

A system with unknown parameters

Instead of the system [Kha96] (Exercise 1.17(4))

$$\left.\begin{aligned} \dot{x_1} &= x_1 + x_2 - x_1\left(|x_1| + |x_2|\right), \\ \dot{x_2} &= -2x_1 + x_2 - x_2\left(|x_1| + |x_2|\right), \end{aligned}\right\} \tag{35}$$

consider an alternative system with eight unknown parameters

$$\left.\begin{aligned} \dot{x_1} &= a_1x_1 + a_2x_2 - x_1\left(a_3|x_1| + a_4|x_2|\right), \\ \dot{x_2} &= a_5x_1 + a_6x_2 - x_2\left(a_7|x_1| + a_8|x_2|\right) + u, \end{aligned}\right\}. \tag{36}$$

Indirect control system with discontinuity

Consider the indirect control system with discontinuity [Lef65] (page 79)

$$\left.\begin{array}{ll} \dot{y} & -ky + \varphi(\sigma) \\ \dot{\sigma} & cy - \rho\varphi(\sigma) \end{array}\right\}, \tag{37}$$

where $k > 0$ and

$$\varphi(\sigma) = \begin{cases} M, & \sigma > 0, \\ -M, & \sigma < 0. \end{cases} \tag{38}$$

There are four cases of response depending on the value of parameters, these four case are

1. $\rho k > c > 0$
2. $c > \rho k > 0$
3. $\rho k < c < 0$
4. $c < \rho k < 0$

Indirec control with a more complicated discontinuity

Consider the same problem described by Equation 37 but here let [Lef65] (page 81)

$$\varphi(\sigma) = \begin{cases} M, & \sigma\alpha > 0, \\ 0, & -\alpha < \sigma < +\alpha, \\ -M, & \sigma < -\alpha. \end{cases} \tag{39}$$

multiple feedback system with switching lines

Consider the system described by [Lef65] (page 85)

$$\left.\begin{array}{ll} \dot{x_1} & = -x_1 + \frac{3}{2}u_1 + \frac{1}{2}u_2 \\ \dot{x_2} & -2x_2 + 2u_1 + 4u_2 \end{array}\right\}, \tag{40}$$

where $u_1, u_2 = \pm 1$

Single-link manipulator with flexible joints

5^{th} **June 1998**

Consider the nonlinear dynamic equations for a single-link manipulator with flexible joints when damping is ignored

$$\begin{aligned} I\ddot{q}_1 + MgL\sin q_1 + k\,(q_1 - q_2) &= 0 && (41) \\ J\ddot{q}_2 - k\,(q_1 - q_2) &= u, && (42) \end{aligned}$$

where I and J are moments of inertia, M is the total mass, g a gravitational constant, k a spring constant, q_1 and q_2 are angular positions and u is a torque input. Let

$$q_1 = x_1, \quad \dot{q}_1 = x_2, \quad q_2 = x_3, \quad \dot{q}_2 = x_4 \tag{43}$$

and obtain the state equations

$$\begin{aligned} \dot{x}_1 &= x_2 && (44) \\ \dot{x}_2 &= -\frac{MgL}{I}\sin x_1 - \frac{k}{I}x_1 + \frac{k}{I}x_3 && (45) \\ \dot{x}_3 &= x_4 && (46) \\ \dot{x}_4 &= \frac{k}{J}x_1 - \frac{k}{J}x_3 + \frac{u}{J}. && (47) \end{aligned}$$

In other words

$$\begin{aligned} \dot{x}_1 &= x_2 && (48) \\ \dot{x}_2 &= a_1\sin x_1 - a_2x_1 + a_2x_3 && (49) \\ \dot{x}_3 &= x_4 && (50) \\ \dot{x}_4 &= a_3x_1 - a_3x_3 + a_4u, && (51) \end{aligned}$$

where $a_1 - \frac{MgL}{I}$, $a_2 = \frac{k}{I}$, $a_3 = \frac{k}{J}$ and $a_4 = \frac{1}{J}$.

For a constant input of magnitude $a = 3$ when $I = 1.3$, $M = 0.4$, $g = 9.8$, $L = 0.8$, $k = 0.5$ and $J = 2$ obtain the result as shown in Figure 3.

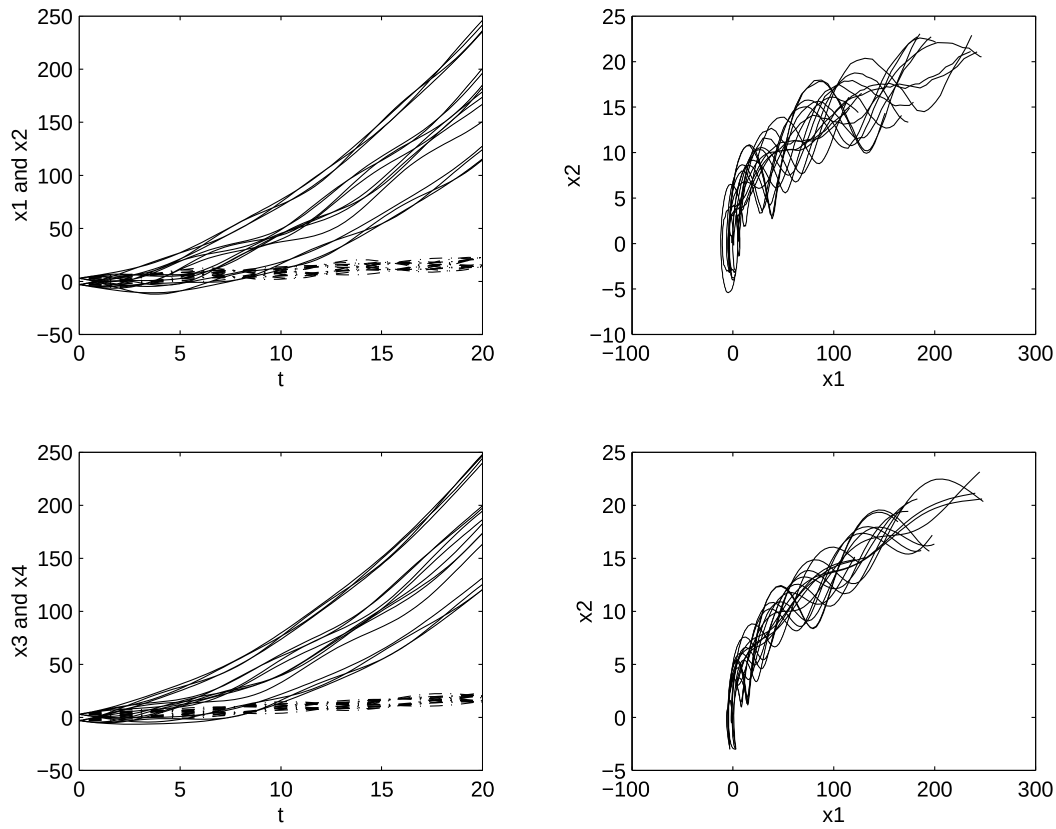

Figure 3: *A single-link manipulator,*$a = 3$*,* $I = 1.3$*,* $M = 0.4$*,* $g = 9.8$*,* $L = 0.8$*,* $k = 0.5$*,* $J = 2.$*,* $t_{CPU} = 0.3100\,\mathrm{sec}$*,* $t_{simulate} = 20\,\mathrm{sec}$*, Initial conditions are all the* $2^4 = 16$ *combinations of* $(x_1, x_2, x_3, x_4) = (\pm 3, \pm 3, \pm 3, \pm 3)$ *. file size = 49.97 KB (*x_2 *and* x_4 *are shown here in dash-and-dot lines.)*

For a step function input of magnitude a

$$u = \begin{cases} a & t \geq 0 \\ 0 & t < 0 \end{cases}, \tag{52}$$

where $a = 3$, $I = 1.3$, $M = 0.4$, $g = 9.8$, $L = 0.8$, $k = 0.5$ and $J = 2$. Obtain the result as shown in Figure 4.

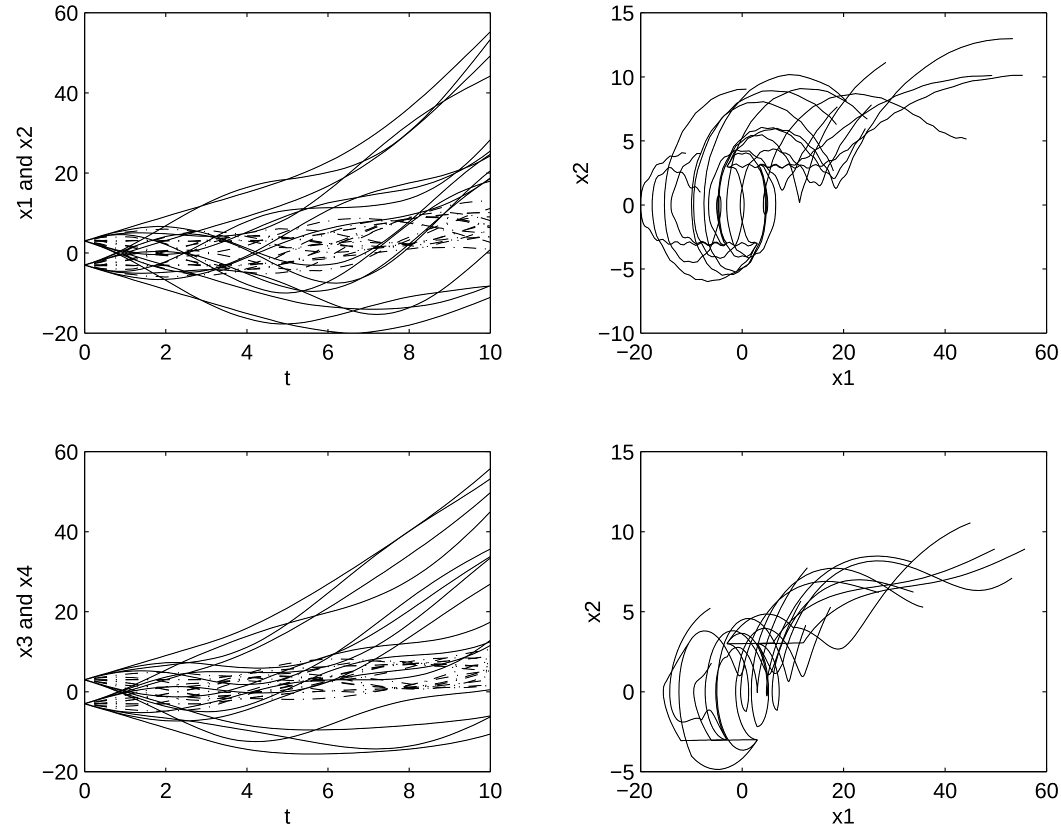

Figure 4: *A single-link manipulator,* $a = 3$, $I = 1.3$, $M = 0.4$, $g = 9.8$, $L = 0.8$, $k = 0.5$, $J = 2.$, $t_{CPU} = 0.2900\,\mathrm{sec}$, $t_{simulate} = 10\,\mathrm{sec}$, *Initial conditions are all the* $2^4 = 16$ *combinations of* $(x_1, x_2, x_3, x_4) = (\pm 3, \pm 3, \pm 3, \pm 3)$ *. file size = 42.29 KB (*x_2 *and* x_4 *are shown here in dash-and-dot lines.)*

For a ramp function input of slope Δ

$$u = \begin{cases} \Delta t & t \geq 0 \\ 0 & t < 0 \end{cases}, \tag{53}$$

where $\Delta = 1.5$, $I = 1.3$, $M = 0.4$, $g = 9.8$, $L = 0.8$, $k = 0.5$ and $J = 2$. Obtain the result as shown in Figure 5.

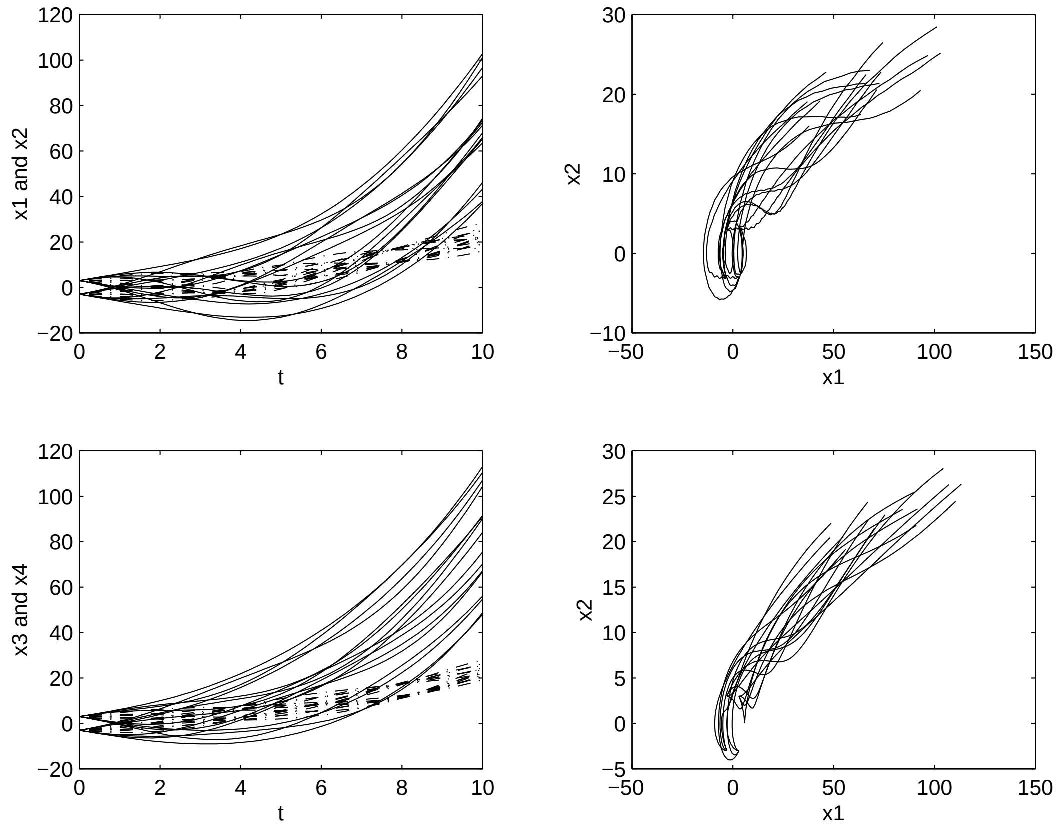

Figure 5: *A single-link manipulator,*$\Delta = 1.5$*,* $I = 1.3$*,* $M = 0.4$*,* $g = 9.8$*,* $L = 0.8$*,* $k = 0.5$*,* $J = 2.$*,* $t_{CPU} = 0.3900\,\mathrm{sec}$*,* $t_{simulate} = 10\,\mathrm{sec}$*, Initial conditions are all the* $2^4 = 16$ *combinations of* $(x_1, x_2, x_3, x_4) = (\pm3, \pm3, \pm3, \pm3)$ *. file size = 43.49 KB (*x_2 *and* x_4 *are shown here in dash-and-dot lines.)*

For a sine function input

$$u = A\sin(\omega t) \tag{54}$$

of amplitude $A = 3$ and frequency $\omega = 2\pi$ when $\Delta = 1.5$, $I = 1.3$, $M = 0.4$, $g = 9.8$, $L = 0.8$, $k = 0.5$ and $J = 2$, obtain the result as shown in Figure 6.

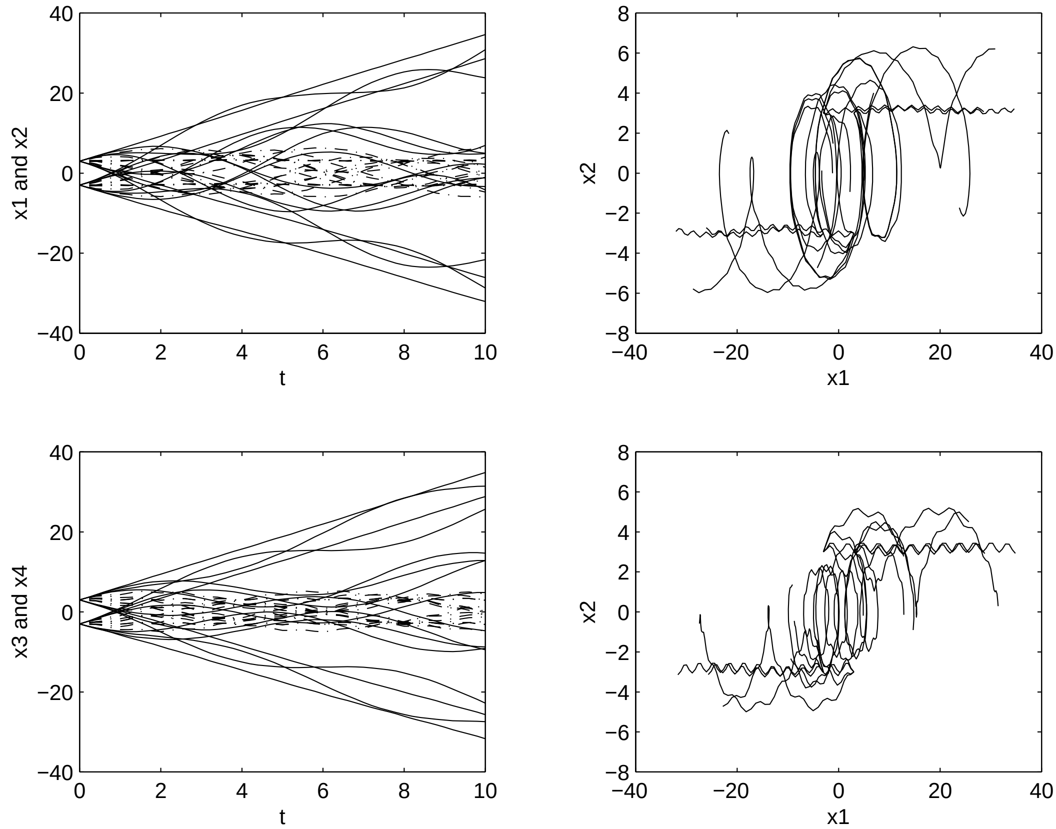

Figure 6: *A single-link manipulator,*= 1.5*,* $I = 1.3$*,* $M = 0.4$*,* $g = 9.8$*,* $L = 0.8$*,* $k = 0.5$*,* $J = 2.$*,* $t_{CPU} = 0.2700\,\mathrm{sec}$*,* $t_{simulate} = 10\,\mathrm{sec}$*, Initial conditions are all the* $2^4 = 16$ *combinations of* $(x_1, x_2, x_3, x_4) = (\pm 3, \pm 3, \pm 3, \pm 3)$ *. file size = 42.12 KB (*x_2 *and* x_4 *are shown here in dash-and-dot lines.)*

Hopfield model

Consider the Hopfield model of the artificial neural network which is based on an RC network connecting amplifiers. The characteristics of the amplifiers are

$$V_i = g_i(u_i), \tag{55}$$

where u_i and V_i are the input and output of voltages of the i^{th} amplifier. $g_i(\cdot)$ is a sigmoid function

$$g_i(\cdot) : \mathcal{R} \rightarrow (-V_M, V_M) \tag{56}$$

with asymptotes $-V_M$ and V_M. some examples of $g_i(\cdot)$ are

$$g_i\left(u_i\right) = \frac{2V_M}{\pi} \tan^{-1} \frac{\lambda \pi u_i}{2V_M}, \quad \lambda > 0 \tag{57}$$

and

$$g_i\left(u_i\right) = V_M \frac{e^{\lambda u_i} - e^{-\lambda u_i}}{e^{\lambda u_i} + e^{-\lambda u_i}}, \quad \lambda > 0, \tag{58}$$

where λ is the slope of $g_i\left(u_i\right)$ at $\left(u_i\right) = 0$.

The Kirchhoff's current law at the input node of the i^{th} amplifier gives

$$C_i \frac{du_i}{dt} = \sum_j T_{ij} V_j - \frac{1}{R_i} u_i + I_i, \tag{59}$$

where

$$\frac{1}{R_i} = \frac{1}{\rho_i} + \sum_j \frac{1}{R_{ij}}, \tag{60}$$

$C_i > 0$ and $\rho_i > 0$ are the total shunt capacitance and shunt resistance at the i^{th} amplifier input, R_{ij} is the resistor connecting the output of the j^{th} amplifier to the i^{th} input line, T_{ij} a signed conductance, $T_{ij} = \frac{1}{R_{ij}}$ or $T_{ij} = -\frac{1}{R_{ij}}$ depending on whether the output of the j^{th} amplifier is positive or negative, and I_i is a constant input current.

By letting the state variables be

$$x_i = V_i, \quad i = 1, 2, \ldots, n \tag{61}$$

we have

$$\begin{aligned} \dot{x}_i &= \frac{dg_i}{du_i}\left(u_i\right) \times \dot{u}_i \qquad (62) \\ &= \frac{dg_i}{du_i}\left(u_i\right) \times \frac{1}{C_i} \left(\sum_j T_{ij} x_j - \frac{1}{R_i} u_i + I_i \right). \qquad (63) \end{aligned}$$

and then

$$\dot{x}_i = \frac{1}{C_i} h_i\left(x_i\right) \left[\sum_j T_{ij} x_j - \frac{1}{R_i} g_i^{-1}\left(x_i\right) + I_i \right], \tag{64}$$

$h_i(x_i)$ being

$$h_i(x_i) \equiv \frac{dg_i}{du_i}(u_i)\bigg|_{u_i=g_i^{-1}(x_i)} \tag{65}$$

Let $n = 2$, $V_M = 1.5$, $T_{21} = T_{12} = 1.2$, $I_i = 0$, $C_i = 0.7$, $\rho_i = 2.1$, $T_{ii} = 0$ and $g_i(u) = \left(\frac{3}{\pi}\right)\tan^{-1}\left(\frac{\lambda\pi u}{2}\right)$, $i = 1, 2$ and $\lambda = 5$.

$$\begin{aligned} h_i(x_i) &= \frac{dg_i}{du}(u) && (66) \\ &= \frac{6\lambda}{4+\lambda^2\pi^2u^2} && (67) \end{aligned}$$

Positioning problem

Consider the positioning problem in [Hoc91] (page 7) but here consider the case where there is an unknown parameter, namely a_1,

$$\dot{x_1} = a_1 x_2 \quad (68)$$

$$\dot{x_2} = u. \quad (69)$$

The control is bound, ie

$$-1 \leq u \leq 1, \quad \forall t. \quad (70)$$

The desired final states are

$$x_1(t_f) = x_2(t_f) = 0. \quad (71)$$

A multiplicative recurrence

Consider a multiplicative recurrence relation described as a discrete system as [Coo86] (Example 7.2)

$$x_1(t+1) = x_2(t) \tag{72}$$
$$x_2(t+1) = x_1(t)x_2(t) \tag{73}$$

Let the Lyapunov function be

$$V(x) = x_1^2 + x_2^2, \tag{74}$$

then

$$\begin{aligned} \Delta V\left(x(t)\right) &= V\left(x(t+1)\right) - V\left(x(t)\right) &(75)\\ &= V\left(f(x)\right) - V(x) &(76)\\ &= x_2^2(t) + x_1^2(t)x_2^2(t) - x_1^2(t) - x_2^2(t) &(77)\\ &= \left(x_2^2(t) - 1\right) x_1^2(t). &(78) \end{aligned}$$

Therefore the domain of attraction is

$$x_1^2 + x_2^2 < 1 \tag{79}$$

and the system is not globally stable.

The simulation result shown in Figure 7 is when initial conditions are inside the domain of attraction while Figure 8 is when they are otherwise.

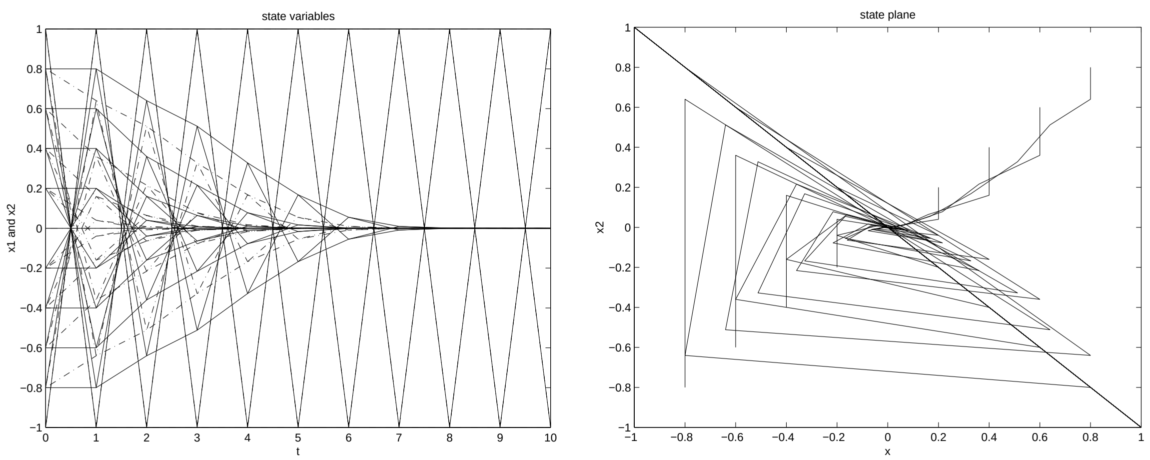

Figure 7: *Multiplicative recurrence, starting points inside the domain of attraction. Initial conditions* $(x_1, x_2) = (i, i), (i, -i), \quad i = -1, -0.8, -0.6, \ldots, 1$

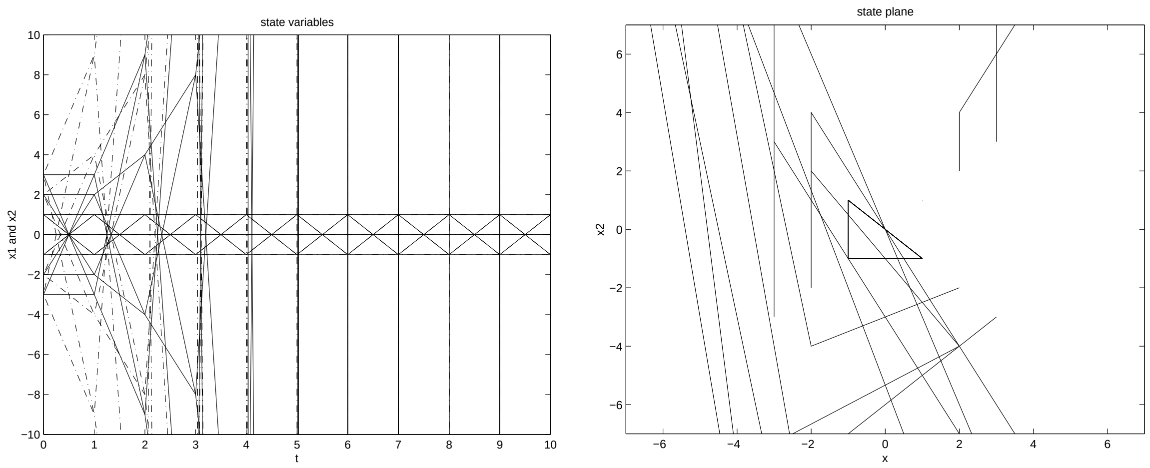

Figure 8: *Multiplicative recurrence, starting points outside the domain of attraction. Initial conditions* $(x_1, x_2) = (i, i), (i, -i), \quad i = -3, -2, -1, \ldots, 3$

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